

Conjugacy Classes of Centralizers in G_2

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Abstract

Let G be an algebraic group of type G_2 over a field k of characteristic $\neq 2, 3$. In this paper we calculate centralizers of semisimple elements in anisotropic G_2 . Using these, we show explicitly that there are six conjugacy classes of centralizers in the compact real form of G_2 .

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1 Introduction

Let G be an algebraic group defined over a field k . Let X be a G -space. Two elements $x, y \in X$ are said to have **same orbit type** if the isotropy subgroups G_x and G_y are conjugate. Let G be a compact Lie group acting on a compact manifold M . It was conjectured by Montgomery (ref. [3], Problem 45) that there are only finitely many orbit types. Floyd proved that if G is a torus group acting on a compact orientable manifold then there are only a finite number of distinct isotropy subgroups (ref. [4], 4.5). Using the results of Floyd, Mostow ([9], Theorem) proved Montgomery's conjecture for a compact Lie group G acting on a compact manifold M . One can consider the action of a group G on itself by conjugation and ask for the orbit types. This case has been considered in [8] by R. Kulkarni where the orbit types are called z -classes. In [13] (Section 3.6 corollary 1 to Theorem 2) Steinberg proved that in a reductive algebraic group G (over algebraically closed field), with $\text{char } G$ good, the number of conjugacy classes of centralizers of elements of G is finite. R. Guraige studied conjugacy classes of centralizers in $M_n(D)$ (the algebra of endomorphisms of a finite dimensional vector space over a central division algebra) in [5].

In this paper we consider the question of calculating conjugacy classes of centralizers of elements for anisotropic groups of type G_2 over a field of characteristic $\neq 2, 3$. Any group of type G_2 over k is precisely the group of k -algebra automorphisms of some octonion algebra over k ([10], Chapter III, Proposition 5, Corollary).

Anisotropic groups of type G_2 over k are given by automorphisms of octonion division algebras over k . We calculate centralizers of elements in G and their conjugacy classes when G is of type G_2 and anisotropic over k . We specifically calculate conjugacy classes of centralizers for compact G_2 (anisotropic group G_2 over \mathbb{R}) and the number of conjugacy classes of centralizers in this case. In particular we show this number is finite (Theorem 4.2). We plan to continue the study of conjugacy classes of centralizers in other groups as well.

2 The Group G_2 and some of its Subgroups

We need description of some subgroups of the group G_2 here. Most of these are described in [11] section 3. Let k be a field and K be an algebraically closed field containing k . We assume that characteristic of $k \neq 2, 3$. Let \mathfrak{C} be an octonion algebra over k . Then the algebraic group $G = \text{Aut}(\mathfrak{C}_K)$ where $\mathfrak{C}_K = \mathfrak{C} \otimes_k K$, is the split, connected, simple algebraic group of type G_2 . Moreover the automorphism group G is defined over k . In fact ([10], Chapter III, Proposition 5, Corollary), any simple group of type G_2 over a field k is isomorphic to the automorphism group of an octonion algebra \mathfrak{C} over k . We abuse notation and continue to write the group G of type G_2 defined over k as $G = \text{Aut}(\mathfrak{C})$ for some k -octonion algebra \mathfrak{C} .

Let $G = \text{Aut}(\mathfrak{C})$ be an anisotropic group of type G_2 , where \mathfrak{C} is an octonion division algebra over k . We fix some notation here. Let \mathfrak{D} be a composition subalgebra of \mathfrak{C} . We denote $G(\mathfrak{C}/\mathfrak{D}) = \{t \in \text{Aut}(\mathfrak{C}) \mid t(x) = x \ \forall x \in \mathfrak{D}\}$ and $G(\mathfrak{C}, \mathfrak{D}) = \{t \in \text{Aut}(\mathfrak{C}) \mid t(x) \in \mathfrak{D} \ \forall x \in \mathfrak{D}\}$, subgroups of G .

Let $\mathfrak{D} = L$ be a quadratic subalgebra. Jacobson described the subgroups $G(\mathfrak{C}/L)$ in his paper (ref. [6]). We mention the description of this subgroup here. Let $L = k(\gamma)$, where $\gamma^2 = c.1 \neq 0$. Then L^\perp is a left L vector space via the octonion multiplication. Also, $h: L^\perp \times L^\perp \longrightarrow L$ defined by $h(x, y) = N(x, y) + c^{-1}\gamma N(\gamma x, y)$ is a non-degenerate hermitian form on L^\perp over L . Any automorphism t of \mathfrak{C} fixing L pointwise induces an L -linear map $t|_{L^\perp}: L^\perp \longrightarrow L^\perp$. Then the group $G(\mathfrak{C}/L)$ is isomorphic to the unimodular unitary group $SU(L^\perp, h)$ of the three dimensional space L^\perp over L relative to the hermitian form h , via the isomorphism, $t \longmapsto t|_{L^\perp}$. We choose $a, b \in \mathfrak{C}$ with $N(a)N(b) \neq 0$ and use Cayley-Dickson doubling process to write $\mathfrak{C} = (L \oplus La) \oplus (L \oplus La)b$. The nontrivial automorphism of L (denoted as $\alpha \mapsto \bar{\alpha}$) can be lifted to an automorphism ρ of \mathfrak{C} with $\rho^2 = 1$ where $\rho(\alpha_1 1 + \alpha_2 a + \alpha_3 b + \alpha_4 ab) = \bar{\alpha}_1 1 + \bar{\alpha}_2 a + \bar{\alpha}_3 b + \bar{\alpha}_4 ab$. Then (see [11], Proposition 3.5) $G(\mathfrak{C}, L) \cong G(\mathfrak{C}/L) \rtimes H$ where $H = \langle \rho \rangle$.

Now we describe subgroups $G(\mathfrak{C}/\mathfrak{D})$ and $G(\mathfrak{C}, \mathfrak{D})$ of $\text{Aut}(\mathfrak{C})$ when $\mathfrak{D} = Q$, a quaternion subalgebra. We denote the norm 1 elements of Q by $SL_1(Q)$. Then we

have, by Cayley-Dickson doubling, $\mathfrak{C} = Q \oplus Qa$ for some $a \in Q^\perp$ with $N(a) \neq 0$. Let $\phi \in \text{Aut}(\mathfrak{C})$ be such that $\phi(x) = x$ for all $x \in Q$. Then for $z = x + ya \in \mathfrak{C}$, there exist $p \in SL_1(Q)$ such that $\phi(z) = x + (py)a$ (ref. [12], Proposition 2.2.1). We denote such an element of $G(\mathfrak{C}/Q)$ by \mathfrak{R}_p . In fact, we have $G(\mathfrak{C}/Q) \cong SL_1(Q)$ via the map $\mathfrak{R}_p \mapsto p$. Now let ψ be an automorphism of Q . Define $\tilde{\psi} \in \text{Aut}(\mathfrak{C})$ by $\tilde{\psi}(x+ya) = \psi(x) + \psi(y)a$. Then one checks easily that $\tilde{\psi}$ is an automorphism of \mathfrak{C} that extends ψ on Q . These automorphisms form a subgroup of $\text{Aut}(\mathfrak{C})$, which, we will abuse notation and, continue to denote by $\text{Aut}(Q)$. We know that any automorphism of a central simple algebra is an inner conjugation by Skolem Noether theorem. For any $c \in Q^*$ we have the automorphism $\mathfrak{I}_c: Q \rightarrow Q$ defined by $x \mapsto cxc^{-1}$. Hence $\text{Aut}(Q) \cong PGL_1(Q) \cong Q^*/k^*$ by $\mathfrak{I}_c \mapsto c$. We then have the subgroup $G(\mathfrak{C}, Q)$ of automorphisms of \mathfrak{C} leaving Q invariant, $G(\mathfrak{C}, Q) \cong G(\mathfrak{C}/Q) \rtimes \text{Aut}(Q)$.

Some of these subgroups are conjugate in the group G . We have Skolem-Noether type theorem for composition algebras ([12], Corollary 1.7.3) which can be used to describe conjugacy of some of these subgroups.

Theorem 2.1 (Skolem Noether type Theorem). *Let \mathfrak{C} be a composition algebra and let \mathfrak{D} and \mathfrak{D}' be subalgebras of the same dimension. Then, every isomorphism from \mathfrak{D} onto \mathfrak{D}' can be extended to an automorphism of \mathfrak{C} .*

We use this theorem to get following,

Proposition 2.2. *Let \mathfrak{C} be an octonion algebra over k and $G = \text{Aut}(\mathfrak{C})$. Let \mathfrak{D} and \mathfrak{D}' be two composition subalgebras of \mathfrak{C} . Suppose \mathfrak{D} and \mathfrak{D}' are isomorphic composition subalgebras. Then the subgroups $G(\mathfrak{C}/\mathfrak{D})$ and $G(\mathfrak{C}/\mathfrak{D}')$ are conjugate in the group G . Also the subgroups $G(\mathfrak{C}, \mathfrak{D})$ and $G(\mathfrak{C}, \mathfrak{D}')$ are conjugate in the group G .*

Proof : Let $\tilde{\phi}$ be the isomorphism of \mathfrak{D} to \mathfrak{D}' . By Theorem 2.1 $\tilde{\phi}$ can be extended to an automorphism of \mathfrak{C} , say ϕ . Then, it is easy to check $\phi G(\mathfrak{C}/\mathfrak{D}) \phi^{-1} = G(\mathfrak{C}/\mathfrak{D}')$ and $\phi G(\mathfrak{C}, \mathfrak{D}) \phi^{-1} = G(\mathfrak{C}, \mathfrak{D}')$. \square

Next we calculate centralizers of elements in the group G and it turns out that they are contained in one of the subgroups described above.

3 Centralizers in Anisotropic G_2

Let G be an anisotropic group of type G_2 defined over a field k . Then there exists \mathfrak{C} , an octonion division algebra over k , such that $G(k) \cong \text{Aut}(\mathfrak{C})$. By abuse of notation we write $G = G(k) = \text{Aut}(\mathfrak{C})$, k -points of G . We fix these notation for this section. We want to calculate centralizer of a semisimple element in G . If k is perfect (e.g., $\text{char}(k) = 0$) every element in G is semisimple ([2], Corollary 8.5). Let $t \in G$. We

denote the centralizer of t in G by $\mathcal{Z}_G(t) = \{g \in G \mid gt = tg\}$. Since G is anisotropic every element of G leaves a subalgebra of \mathfrak{C} fixed pointwise ([11], Lemma 6.1). We denote the subalgebra of fixed points of t by $\mathfrak{C}^t = \{x \in \mathfrak{C} \mid t(x) = x\}$. We note that t restricted to \mathfrak{C}_0 , trace zero space of \mathfrak{C} , is an element of special orthogonal group of an odd dimensional space ([12], Proposition 2.2.2), hence t has a fixed point in \mathfrak{C}_0 by Cartan-Dieudonne theorem. This implies that the dimension of \mathfrak{C}^t is ≥ 2 . As the dimension of a composition subalgebra can be 2, 4 or 8 the subalgebra \mathfrak{C}^t is either a quadratic field extension of k , a quaternion subalgebra or $t = I$. Hence we need to calculate centralizers of elements which are contained in a subgroup $G(\mathfrak{C}/L)$ where L is a quadratic field extension or $G(\mathfrak{C}/Q)$ where Q is a quaternion subalgebra. However we have,

Proposition 3.1. *Let G be an anisotropic group of type G_2 over k . Let \mathfrak{C} be the octonion division algebra over k such that $G = \text{Aut}(\mathfrak{C})$. Let $t \in G$. Then \mathfrak{C}^t is a composition subalgebra of \mathfrak{C} and the centralizer $\mathcal{Z}_G(t) \subset G(\mathfrak{C}, \mathfrak{C}^t)$.*

Proof : The subalgebra $\mathfrak{C}^t \subset \mathfrak{C}$ is a composition subalgebra as \mathfrak{C} is division. Let $g \in \mathcal{Z}_G(t)$. Then,

$$t(g(x)) = g(x), \forall x \in \mathfrak{C}^t.$$

Hence $g(x) \in \mathfrak{C}^t, \forall x \in \mathfrak{C}^t$. This shows that $g \in G(\mathfrak{C}, \mathfrak{C}^t)$ and $\mathcal{Z}_G(t) \subset G(\mathfrak{C}, \mathfrak{C}^t)$. \square

3.1 \mathfrak{C}^t is a Quaternion Algebra

Let $\mathfrak{C}^t = Q$ be a quaternion subalgebra of the octonion division algebra \mathfrak{C} , hence Q itself is division.

Lemma 3.2. *With notation as above, let $t = \mathfrak{R}_p \in G(\mathfrak{C}/Q)$ for some $p \in SL_1(Q)$ with $p \notin k$. Then $\mathcal{Z}_G(t) = \{\mathfrak{R}_{p_1} \mathfrak{I}_{c_1} \in G(\mathfrak{C}, Q) \mid p_1 c_1 \in L\}$ where $L = k(p)$, a quadratic field extension of k .*

Proof : We write $\mathfrak{C} = Q \oplus Qb$ for some $b \in Q^\perp$. Then $t(x + yb) = x + (py)b$. From Proposition 3.1 we get, $\mathcal{Z}_G(t) \subset G(\mathfrak{C}, Q) = \{\mathfrak{R}_{p_1} \mathfrak{I}_{c_1} \mid p_1 \in SL_1(Q), c_1 \in Q^*\}$. Let $g \in \mathcal{Z}_G(t)$ and let $g = \mathfrak{R}_{p_1} \mathfrak{I}_{c_1}$. Then

$$gt = tg \Rightarrow \mathfrak{R}_{p_1} \mathfrak{I}_{c_1} R_p = \mathfrak{R}_p \mathfrak{R}_{p_1} \mathfrak{I}_{c_1} \Rightarrow \mathfrak{R}_{p_1 c_1 p c_1^{-1}} \mathfrak{I}_{c_1} = \mathfrak{R}_{pp_1} \mathfrak{I}_{c_1}$$

and we get, $p_1 c_1 p = p p_1 c_1$, i.e., $p_1 c_1 \in \mathcal{Z}_Q(p) = L$ where $L = k(p)$ is a quadratic field extension of k . Hence $\mathcal{Z}_G(t) = \{\mathfrak{R}_{p_1} \mathfrak{I}_{c_1} \mid p_1 c_1 \in L, c_1 \in Q^*, p_1 \in SL_1(Q)\}$. \square

Now we consider conjugacy classes of centralizers of these elements in G .

Lemma 3.3. *Let $t, t' \in G = \text{Aut}(\mathfrak{C})$. Let t and t' leave quaternion subalgebra Q and Q' fixed pointwise, respectively. Suppose Q and Q' are isomorphic. Let $t = \mathfrak{R}_p$ and $t' = \mathfrak{R}_{p'}$ where $p \in SL_1(Q)$ and $p' \in SL_1(Q')$ and both $p, p' \notin k$. Suppose $L = k(p)$ and $L' = k(p')$ are isomorphic field extensions of k . Then $\mathcal{Z}_G(t)$ is conjugate to $\mathcal{Z}_G(t')$ in G .*

Proof : As Q and Q' are isomorphic, by Theorem 2.1 we have an automorphism ϕ of \mathfrak{C} such that $\phi|_Q$ is the given isomorphism of Q to Q' . By conjugating the element t' by ϕ we may assume t and t' both belong to $G(\mathfrak{C}/Q)$. Let L and L' be isomorphic. Then there exists an isomorphism $\psi = \mathfrak{I}_c$, conjugation by $c \in Q$, which gives the isomorphism of L to L' . Let $g \in \mathcal{Z}_G(\mathfrak{R}_p)$. From previous lemma $g = \mathfrak{R}_{p_1} \mathfrak{I}_{c_1}$ with $p_1 c_1 \in L$. Then,

$$\psi g \psi^{-1} = \mathfrak{I}_c \mathfrak{R}_{p_1} \mathfrak{I}_{c_1} \mathfrak{I}_{c^{-1}} = \mathfrak{R}_{cp_1 c^{-1}} \mathfrak{I}_{cc_1 c^{-1}}.$$

We note that $cp_1 c^{-1} cc_1 c^{-1} = cp_1 c_1 c^{-1} \in L'$. This implies $\psi g \psi^{-1} \in \mathcal{Z}_G(t')$ and hence $\mathcal{Z}_G(t)$ is conjugate to $\mathcal{Z}_G(t')$ in G . \square

We note that the center of $SL_1(Q)$ is $\{1, -1\}$ and the element t in $\text{Aut}(\mathfrak{C})$ corresponding to -1 is a non-trivial involution (i.e. $t^2 = 1$). By a similar calculation as above it is easy to see that $\mathcal{Z}_G(t) = G(\mathfrak{C}, Q)$ for $1 \neq t$ an involution. In fact any non-trivial involution in $\text{Aut}(\mathfrak{C})$ correspond to a quaternion subalgebra in this fashion. Two involutions are conjugate if and only if the corresponding fixed quaternion subalgebras are isomorphic ([11], Section 4). From Proposition 2.2, two involutions have their centralizers conjugate if and only if the corresponding fixed quaternion subalgebras are isomorphic. We observe that the centralizers corresponding to involutions and other type of elements in Lemma 3.3 are not isomorphic hence they can not be conjugate in the group G .

3.2 \mathfrak{C}^t is a Quadratic Field Extension

Now suppose $\mathfrak{C}^t = L$, a quadratic field extension of k and $t \in G(\mathfrak{C}/L)$. We denote the non-trivial automorphism of L by $\alpha \mapsto \bar{\alpha}$. Moreover we can find $a, b \in \mathfrak{C}$ such that $Q = L \oplus La$ is a quaternion subalgebra and $\mathfrak{C} = Q \oplus Qb$. Then L^\perp is a hermitian space over L . With respect to the basis $\{a, b, c = ab\}$ we write the subgroup $SU(L^\perp, h)$ of G as $SU(H) = \{A \in SL(3, L) \mid {}^t A H \bar{A} = H\}$ where $H = \text{diag}\{h(a, a), h(b, b), h(ab, ab)\}$. We denote the matrix of t as A with respect to this basis. We observe that t leaves a quaternion subalgebra fixed pointwise if and only if 1 is an eigenvalue of A , i.e., $X - 1$ is a factor of the characteristic polynomial $\chi_A(X)$. Since $\mathfrak{C}^t = L$, we may assume that $X - 1$ is not a factor of $\chi_A(X)$. We have,

Lemma 3.4. *With notation as above, let $t \in G(\mathfrak{C}/L)$. Suppose 1 is not a root of $\chi_A(X)$. Then, $\mathcal{Z}_G(t) \subset G(\mathfrak{C}/L)$.*

Proof : Let $t \in G(\mathfrak{C}/L)$ be such that t does not fix any point in L^\perp . Let $g \in \mathcal{Z}_G(t)$ then $g \in G(\mathfrak{C}, L) \cong SU(L^\perp, h) \rtimes \langle \rho \rangle$ where ρ is an extension of conjugation on L ([11], Proposition 3.5). We denote the matrix of t in $SU(H)$ by A . Then, either $g \in SU(L^\perp, h)$ or $g = h\rho$ for some $h \in SU(L^\perp, h)$. Suppose $g = h\rho$. We denote the matrix of h by B and we denote the action of ρ on elements of L by $\alpha \mapsto \bar{\alpha}$. We claim that in this case there is no such element in $\mathcal{Z}_G(t)$. We have,

$$\begin{aligned} gt(\alpha_1 a + \alpha_2 b + \alpha_3 c) &= h\rho t(\alpha_1 a + \alpha_2 b + \alpha_3 c) = h\rho(\alpha_1 Aa + \alpha_2 Ab + \alpha_3 Ac) \\ &= h(\bar{\alpha}_1 \bar{A}a + \bar{\alpha}_2 \bar{A}b + \bar{\alpha}_3 \bar{A}c) = \bar{\alpha}_1 B\bar{A}a + \bar{\alpha}_2 B\bar{A}b + \bar{\alpha}_3 B\bar{A}c \end{aligned}$$

and

$$\begin{aligned} tg(\alpha_1 a + \alpha_2 b + \alpha_3 c) &= th\rho(\alpha_1 a + \alpha_2 b + \alpha_3 c) = th(\bar{\alpha}_1 a + \bar{\alpha}_2 b + \bar{\alpha}_3 c) \\ &= t(\bar{\alpha}_1 Ba + \bar{\alpha}_2 Bb + \bar{\alpha}_3 Bc) = \bar{\alpha}_1 ABa + \bar{\alpha}_2 ABb + \bar{\alpha}_3 ABc. \end{aligned}$$

As $g \in \mathcal{Z}_G(t)$, above calculation implies $AB = B\bar{A}$ where $B \in SU(H)$. In this case the characteristic polynomial $\chi_A(X) = X^3 - aX^2 + \bar{a}X - 1$ will have property $\chi_A(X) = \chi_{\bar{A}}(X) = \overline{\chi_A(X)}$ which implies $a = \bar{a}$. But $\chi_A(X) = X^3 - aX^2 + aX - 1 = (X-1)(X^2 + (1-a)X + 1)$ is reducible and A has 1 as an eigenvalue. Which contradicts the assumption that 1 is not a root of $\chi_A(X)$. Hence $\mathcal{Z}_G(t) \subset G(\mathfrak{C}/L)$. \square

Let $t \in G(\mathfrak{C}/L)$ and let $A \in SU(H)$ be the corresponding matrix. Moreover we assume that 1 is not a root of $\chi_A(X)$. Suppose t is semisimple, we have the following cases:

1. (a) The characteristic polynomial of A is irreducible over L .
 (b) The characteristic polynomial has form $(X - \alpha)f(X)$ where $f(X)$ is a quadratic irreducible over L .
 (c) The characteristic polynomial of A has distinct roots over L , i.e., $\chi_A(X) = (X - \alpha)(X - \beta)(X - \gamma)$.
2. The characteristic polynomial of A has two distinct roots over L , i.e., $\chi_A(X) = (X - \alpha)(X - \beta)^2$ and minimal polynomial is $(X - \alpha)(X - \beta)$ where $\bar{\alpha} = \alpha^{-1}$ and $\bar{\beta} = \beta^{-1}$.
3. The characteristic polynomial of A is $\chi_A(X) = (X - \alpha)^3$ and the minimal polynomial is $(X - \alpha)$ where $\bar{\alpha} = \alpha^{-1}$.

We calculate centralizers in these cases. We observe that whenever H is a connected subgroup of a connected algebraic group G and $t \in H$ such that $\mathcal{Z}_G(t) \subset H$ then t is regular (or strongly regular) in G if and only if it is so in H . Hence $t \in G(\mathfrak{C}/L)$

with $\mathfrak{C}^t = L$ is regular in $G(\mathfrak{C}/L)$ if and only if it is regular in G (Lemma 3.4). This implies that any element of $G(\mathfrak{C}/L)$ (which does not have 1 as an eigenvalue) is a regular element of G in the case 1, i.e., the connected component of centralizers of such elements are maximal tori. In the case 3 the element is scalar $A = \alpha.Id$ and in this case centralizer is whole of $SU(H)$. We calculate the centralizer in case 2 below.

Lemma 3.5. *Let $t \in G = \text{Aut}(\mathfrak{C})$. With notations as above let $t \in G(\mathfrak{C}/L)$ and A be the corresponding element in $SU(H)$. Suppose 1 is not an eigenvalue of A and $\chi_A(X) = (X - \alpha)(X - \beta)^2$ and minimal polynomial is $(X - \alpha)(X - \beta)$ where $\bar{\alpha} = \alpha^{-1}$ and $\bar{\beta} = \beta^{-1}$. Then the centralizer of t is isomorphic to a subgroup $\left\{ \begin{pmatrix} \det(S)^{-1} & 0 \\ 0 & S \end{pmatrix} \in SU(H) \mid S \in U(W, h|_W) \right\}$ for some 2-dimensional nondegenerate L -subspace W of L^\perp .*

Proof : We have $\chi_A(X) = (X - \alpha)(X - \beta)^2$ where $\alpha, \beta \in L$ both not equal to 1 and the minimal polynomial is $(X - \alpha)(X - \beta)$. Then we can choose a basis of L^\perp consisting of eigenvectors v_1, v_2, v_3 corresponding to eigenvalues α, β, β of t . We denote the subspace generated by v_2, v_3 as W . The matrix of t is diagonal with respect to this basis, $A = \text{diag}\{\alpha, \beta, \beta\}$. As 1 is not a root of $\chi_A(X)$ the centralizer $\mathcal{Z}_G(t) \cong \mathcal{Z}_{SU(H)}(A)$ which is isomorphic to $\mathcal{Z}_{GL_3(L)}(A) \cap SU(H) = \left\{ \begin{pmatrix} \det(S)^{-1} & 0 \\ 0 & S \end{pmatrix} \mid S \in U(W, h|_W) \right\}$. \square

4 Conjugacy Classes of Centralizers in Compact G_2

Let G be the compact real group of type G_2 . Then (ref. [1] Chapter V, Section 24.6 Statement (ii) under (c)) G is \mathbb{R} -points of an anisotropic group of type G_2 . We consider the unique anisotropic form of G_2 over \mathbb{R} . As there is a unique anisotropic Pfister form over \mathbb{R} of dimension 8, there is a unique octonion division algebra (up to isomorphism) \mathfrak{C} over \mathbb{R} and $G(\mathbb{R}) \cong \text{Aut}(\mathfrak{C})$. We calculate centralizers of elements and there conjugacy classes in this case. Let \mathfrak{C} be the octonion division algebra over \mathbb{R} and we write, by abuse of notation, $G = G(\mathbb{R}) = \text{Aut}(\mathfrak{C})$. Let $t \in G$. Let L be a quadratic field extension of \mathbb{R} left fixed pointwise by t (which is isomorphic to \mathbb{C}). Then $t \in SU(L^\perp, h) \cong SU(3) = \{A \in GL_3(\mathbb{C}) \mid {}^t A \bar{A} = 1\}$. We also note that all quadratic field extensions of \mathbb{R} contained in \mathfrak{C} are isomorphic hence all subgroups of type $SU(L^\perp, h)$ are conjugate (Proposition 2.2) in G . We denote the non-trivial automorphism of L (the complex conjugation in case of \mathbb{C}) by $\alpha \mapsto \bar{\alpha}$. We note that every element in this case, $k = \mathbb{R}$, is semisimple in G . Also every element of $SU(3)$

can be diagonalized in $SU(3)$. Hence any element $A \in SU(3)$ is of one of the following types:

- (a) The characteristic polynomial of A is $\chi_A(X) = (X - \alpha)(X - \beta)(X - \gamma)$ where α, β and γ are distinct.
- (b) The characteristic polynomial of A is $\chi_A(X) = (X - \alpha)(X - \beta)^2$ and the minimal polynomial is $(X - \alpha)(X - \beta)$ where $\alpha\bar{\alpha} = 1 = \beta\bar{\beta}$
- (c) The characteristic polynomial of A is $\chi_A(X) = (X - \alpha)^3$ and the minimal polynomial is $(X - \alpha)$ where $\alpha\bar{\alpha} = 1$.

If A has three distinct roots and none of them is 1 then by Lemma 3.4 the centralizer of t is contained in $G(\mathfrak{C}/L)$. A simple calculation shows that the centralizer of A in $SU(3)$ is a maximal torus. Since all maximal tori of G are conjugate (ref. [7], Chapter IV, Corollary 4.35) and they are contained in a subgroup isomorphic to $SU(3)$, we get that the element t is strongly regular (those elements of which centralizer is a maximal torus) in G . Moreover the centralizers form one conjugacy class. If 1 is an eigenvalue the centralizer may not be contained in $G(\mathfrak{C}/L)$. In fact we have,

Lemma 4.1. *With notation as above, let $t \in G(\mathfrak{C}/L) \subset G$ where $L \subset \mathfrak{C}$ is a quadratic field extension of \mathbb{R} . In addition, we assume that the characteristic polynomial of t has distinct roots. Then, $\mathcal{Z}_G(t) \subset G(\mathfrak{C}/L)$ if and only if t does not leave any quaternion subalgebra fixed pointwise. Moreover, if t leaves a quaternion algebra fixed pointwise we have $\mathcal{Z}_G(t) \cong \mathcal{Z}_{G(\mathfrak{C}/L)}(t) \rtimes \mathbb{Z}/2\mathbb{Z}$.*

Proof : If $t \in G(\mathfrak{C}/L)$ does not leave any quaternion subalgebra fixed pointwise then the characteristic polynomial of t does not have 1 as a root. Hence from Lemma 3.4 it follows that $\mathcal{Z}_G(t) \subset G(\mathfrak{C}/L)$. Now suppose $t \in G(\mathfrak{C}/L)$ leaves a quaternion subalgebra Q fixed pointwise. By using Theorem 2.1 we may assume Q contains L . As $t \in G(\mathfrak{C}/L) \cong SU(3)$ the corresponding matrix A can be diagonalized in the subgroup $SU(3)$. We write the matrix of t as $A = \text{diag}\{1, \alpha, \bar{\alpha}\}$ for some $\alpha \in L$ with $\alpha\bar{\alpha} = 1$. Note that $\alpha \neq \bar{\alpha}$ else $\alpha = \pm 1$ and the roots are not distinct. We check that $\rho \in G(\mathfrak{C}, L)$ conjugates t to t^{-1} (note that $\bar{A} = A^{-1}$). We also find an element

$B \in SU(3)$ such that $AB = B\bar{A}$. We take $B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ then one can check

that $B \in SU(3)$ and $AB = B\bar{A}$. Combined together we get an element $g = h\rho$ such that $g \in \mathcal{Z}_G(t)$ and $g^2 = 1$ where $h \in G(\mathfrak{C}/L)$ corresponds to the element $B \in SU(3)$. We check that $\mathcal{Z}_G(t) \cong \mathcal{Z}_{G(\mathfrak{C}/L)}(t) \rtimes \langle g \rangle$. It is also clear that $\mathcal{Z}_G(t)$ is not contained in $G(\mathfrak{C}/L)$. \square

If t has 1 as an eigenvalue the centralizer $\mathcal{Z}_G(t) \cong T \rtimes \mathbb{Z}/2\mathbb{Z}$ where T is a maximal torus in G (which is always contained in a subgroup $SU(3)$).

In the second case with respect to the basis chosen as above $A = \text{diag}\{\alpha, \beta, \beta\}$. Suppose neither of α, β is 1 then the centralizer is (ref. Lemma 3.5)

$$\mathcal{Z}_G(t) \cong \mathcal{Z}_{SU(3)}(A) \cong \left\{ \begin{pmatrix} \det(S)^{-1} & 0 \\ 0 & S \end{pmatrix} \mid S \in U(2) \right\}.$$

If 1 is an eigenvalue we have $A = \text{diag}\{1, -1, -1\}$. In this case t is an involution and the centralizer is contained in $G(\mathfrak{C}, L)$. We note that ρ also centralizes t and hence $\mathcal{Z}_G(t) \cong \mathcal{Z}_{G(\mathfrak{C}/L)}(t) \rtimes \langle \rho \rangle \cong U(2) \rtimes \mathbb{Z}/2\mathbb{Z}$.

In the third case if $\alpha \neq 1$ centralizer is whole of $SU(3)$. If $\alpha = 1$ the centralizer is whole G .

For simplicity we assume $L = \mathbb{C}$ and make a table of representative elements of each orbit type here.

representative element of an orbit type	corresponding centralizer
I	G
$\text{diag}(e^{-i(\theta+\phi)}, e^{i\theta}, e^{i\phi})$ $e^{i\theta}, e^{i\phi} \neq \pm 1, \pm i$	maximal torus of G
$\text{diag}(1, e^{i\theta}, e^{-i\theta}); e^{i\theta} \neq \pm 1$	$T \rtimes \mathbb{Z}/2\mathbb{Z}$, T a maximal torus
$\text{diag}(e^{-2i\theta}, e^{i\theta}, e^{i\theta}); e^{i\theta} \neq \pm 1$	$U(2)$
$\text{diag}(1, -1, -1)$	$U(2) \rtimes \mathbb{Z}/2\mathbb{Z}$
$\text{diag}(\omega, \omega, \omega); \omega \neq 1, \omega^3 = 1$	$SU(3)$

Hence we have proved,

Theorem 4.2. *Let G be the anisotropic group of type G_2 over \mathbb{R} . Then there are exactly six orbit types (conjugacy classes of centralizers) as displayed in the table above.*

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References

- [1] A. Borel, Linear algebraic groups. Graduate Texts in Mathematics, 126. Springer-Verlag, New York, 1991.
- [2] A. Borel and J. Tits, Groupes réductifs. *Inst. Hautes Études Sci. Publ. Math.* No. 27 1965 55–150.

- [3] S. Eilenberg, On the problems of topology, *Ann. of Math.* (2) 50 (1949) 247–260.
- [4] E. E. Floyd, Orbits of torus groups operating on manifolds, *Ann. of Math.* (2) 65 (1957) 505–512.
- [5] R. Gouraige, z -classes of Elements in Central Simple Algebras, *Ph.D. thesis submitted at City University of New York* (April 2006).
- [6] N. Jacobson, Composition Algebras and Their Automorphisms, *Rend. Circ. Mat. Palermo* (2) (1958) 55–80.
- [7] A. W. Knap, Lie groups beyond an introduction. Progress in Mathematics, 140. *Birkhuser Boston, Inc., Boston, MA*, 2002.
- [8] R. S. Kulkarni, Dynamical types and conjugacy classes of centralizers in groups, *J. Ramanujan Math. Soc.* 22 (2007), no. 1, 35–56.
- [9] G. D. Mostow, On a conjecture of Montgomery, *Ann. of Math.* (2) 65 (1957) 513–516.
- [10] J. P. Serre, Galois cohomology, *Springer-Verlag*, Berlin, (1997).
- [11] A. Singh and M. Thakur, Reality Properties of Conjugacy Classes in G_2 , *Israel Journal of Mathematics* 145 (2005) 157–192.
- [12] T. A. Springer and F. D. Veldkamp, Octonions, Jordan Algebras and Exceptional Groups, *Springer Monographs in Mathematics*, Springer-Verlag, Berlin, (2000).
- [13] R. Steinberg, Conjugacy classes in algebraic groups. Notes by Vinay V. Deodhar. *Lecture Notes in Mathematics*, Vol. 366. Springer-Verlag, Berlin-New York, 1974.
- [14] M. J. Wonenburger, Automorphisms of Cayley Algebras, *J. Algebra* 12 (1969) 441–452.

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